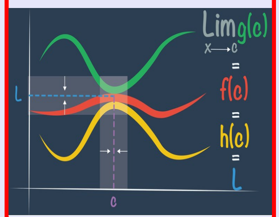
Calculus I

Lecture 51



Feb 19-8:47 AM

Find f_{ave} for $f(x) = \sin 4x$ over $[-\pi, \pi]$

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$f_{ave} = \frac{1}{\pi-(-\pi)}\int_{-\pi}^{\pi} \sin 4x\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin 4x\,dx$$

$u = 4x$ $\qquad$ $x = -\pi \rightarrow u = -4\pi$

$du = 4\,dx$ $\qquad$ $x = \pi \rightarrow u = 4\pi$

$\frac{du}{4} = dx$

$$= \frac{1}{2\pi}\int_{-4\pi}^{4\pi} \sin u\,\frac{du}{4} = \frac{1}{2\pi}\cdot\frac{1}{4}\int_{-4\pi}^{4\pi} \sin u\,du$$

$$= \frac{1}{8\pi}\cdot -\cos u\Big|_{-4\pi}^{4\pi}$$

$$= \frac{-1}{8\pi}\left[\underbrace{\cos 4\pi}_{1} - \underbrace{\cos(-4\pi)}_{1}\right]$$

$$= \frac{-1}{8\pi}[1-1] = \boxed{0}$$

May 16-8:46 AM

Find f_{ave} for $f(x) = x^2(1+x^3)^4$ over $[0,2]$.

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$f_{ave} = \frac{1}{2-0}\int_0^2 x^2(1+x^3)^4\,dx$$

$$= \frac{1}{2}\int_0^2 x^2(1+x^3)^4\,dx$$

$u = 1+x^3$ $\quad x=0 \to u=1$

$du = 3x^2\,dx$ $\quad x=2 \to u=9$

$\frac{du}{3} = x^2\,dx$

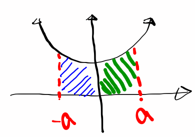
$$= \frac{1}{2}\int_1^9 u^4\,\frac{du}{3} = \frac{1}{6}\cdot\frac{u^5}{5}\Big|_1^9 = \frac{1}{30}\left[9^5 - 1^5\right]$$

$$= \frac{59048}{30} = \boxed{\frac{29524}{15}}$$

May 16-8:52 AM

If $f(x)$ is an even function,

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$



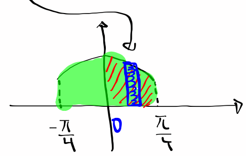
even function

Ex: $\int_{-2}^{2} x^4\,dx = 2\int_0^2 x^4\,dx$

$= 2\cdot\frac{x^5}{5}\Big|_0^2 = \boxed{\frac{64}{5}}$

Find the area below $f(x) = \cos x$, above x-axis, from $x = \frac{-\pi}{4}$ to $x = \frac{\pi}{4}$.

$\cos x$ is an even function



$$A = \int_{-\frac{\pi}{4}}^{\pi/4} \cos x\,dx = 2\int_0^{\pi/4} \cos x\,dx = 2\cdot \sin x\Big|_0^{\pi/4}$$

$$= 2\left[\sin\frac{\pi}{4} - \sin 0\right]$$

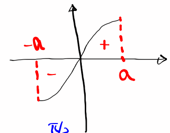
$$= 2\left[\frac{\sqrt{2}}{2} - 0\right]$$

$$= \boxed{\sqrt{2}}$$

May 16-9:00 AM

If $f(x)$ is an odd function,

$$\int_{-a}^{a} f(x)\,dx = 0 \checkmark$$



$$\int_{-\pi/2}^{\pi/2} \sin x \; dx = 0$$

$\sin x$ is an odd function

$$\int_{-\pi/2}^{\pi/2} \sin x \, dx = -\cos x \Big|_{-\pi/2}^{\pi/2}$$

$$= -\left[\cos\frac{\pi}{2} - \cos\left(-\frac{\pi}{2}\right)\right]$$

$$= -0 = \boxed{0}$$

Evaluate $\int_{-1}^{1} \sqrt[3]{x}\, dx = \boxed{0}$

$f(x) = \sqrt[3]{x}$

$f(-x) = \sqrt[3]{-x}$

$= -\sqrt[3]{x} = -f(x)$

So $f(x) = \sqrt[3]{x}$ is an odd function

$$\int_{-1}^{1} \sqrt[3]{x}\, dx = \int_{-1}^{1} x^{1/3} dx = \frac{x^{1/3+1}}{1/3+1}\Big|_{-1}^{1}$$

$$= \frac{x^{4/3}}{\frac{4}{3}}\Big|_{-1}^{1} = \frac{3}{4}\left[1^{4/3} - (-1)^{4/3}\right]$$

$$= \frac{3}{4}\cdot 0 = \boxed{0}$$

$f(x) = |x|$

$f(-x) = |-x| = |-1||x| = |x| = f(x)$

$\therefore$ Even function

$f(x) = \sin x$ is an odd function

$f(x) = \sin x^2$ " " even function.

$$\boxed{\int_{a}^{a} f(x)\,dx = 0}$$

May 16-9:07 AM

Mean - Value Theorem for differentiation

1) $f(x)$ is cont. on $[a,b]$

2) $f(x)$ is diff. on (a,b)

then there is a c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Mean - Value theorem for integration:

If $f(x)$ is cont. on $[a,b]$, then there is a c in $[a,b]$ such that

$$f(c) = f_{ave} = \frac{1}{b-a}\int_{a}^{b} f(x)\,dx$$

May 16-9:19 AM

Suppose $f(x) = \sqrt{x}$ and $[0,4]$

By MVT for integration, there is a number c in $[0,4]$ such that $f(c) = f_{ave}$

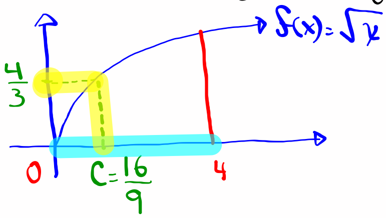
$$\sqrt{c} = \frac{1}{4-0}\int_0^4 \sqrt{x}\,dx \rightarrow x^{1/2}$$

$$\sqrt{c} = \frac{1}{4} \cdot \frac{x^{3/2}}{3/2}\Big|_0^4$$

$$\sqrt{c} = \frac{1}{6}\, x\sqrt{x}\Big|_0^4 \rightarrow \sqrt{c} = \frac{1}{6}\left[4\sqrt{4} - 0\right]$$

$$\sqrt{c} = \frac{1}{6}[8]$$

$$\sqrt{c} = \frac{4}{3} \qquad \boxed{c = \frac{16}{9}}$$



May 16-9:23 AM

Suppose $f(x) = x^2 + 2$ over $[-1,3]$

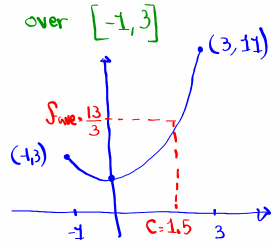
$f(-1) = 3$

$f(3) = 11$

$f(x)$ is cont. on $[-1,3]$

$f(x)$ is a parabola

Vertex at $(0,2)$



By MVT for integration, there is a number c in $[-1,3]$ such that $f(c) = f_{ave}$.

$$c^2 + 2 = \frac{1}{3-(-1)}\int_{-1}^{3} (x^2+2)\,dx$$

$$c^2 + 2 = \frac{1}{4}\left[\frac{x^3}{3} + 2x\right]\Big|_{-1}^{3}$$

$$c^2 + 2 = \frac{1}{4}\left[\left(\frac{3^3}{3} + 2(3)\right) - \left(\frac{(-1)^3}{3} + 2(-1)\right)\right]$$

$$c^2 + 2 = \frac{1}{4}\left[9 + 6 + \frac{1}{3} + 2\right] \rightarrow c^2 + 2 = \frac{1}{4}\left[17\tfrac{1}{3}\right]$$

$$c^2 + 2 = \frac{1}{\cancel{4}} \cdot \frac{\cancel{52}^{13}}{3}$$

$$c^2 = \frac{13}{3} - 2$$

$$c^2 = \frac{7}{3} \qquad c = \pm\sqrt{\frac{7}{3}}$$

$$c \approx \pm 1.5$$

May 16-9:31 AM

Suppose $f(x) = \int_{u(x)}^{v(x)} g(t)\,dt$

$$f'(x) = g(v(x))\cdot v'(x) - g(u(x))\cdot u'(x)$$

$f(x) = \int_{2x}^{\overbrace{x^2}^{v(x)}} \sin t\,dt$

$$\boxed{f'(x) = \sin x^2 \cdot 2x - \sin 2x \cdot 2}$$

$$\boxed{f'(x) = 2x\sin x^2 - 2\sin 2x}$$

Find $\frac{d}{dx}\int_{\underbrace{1}_{u(x)}}^{x^5} \underbrace{\sec t}_{g(t)}\,dt = \sec x^5 \cdot 5x^4 - \sec 1 \cdot 0$

$$= \boxed{5x^4 \sec x^5}$$

May 16-9:41 AM

Find $\frac{d}{dx}\int_{\underbrace{\sqrt{x}}_{u(x)}}^{\overbrace{x^3}^{v(x)}} \underbrace{\cos t^2}_{g(t)}\,dt$

$$= \underbrace{\cos((x^3)^2)}_{g(v(x))}\cdot \underbrace{3x^2}_{v'(x)} - \underbrace{\cos((\sqrt{x})^2)}_{g(u(x))}\cdot \underbrace{\frac{1}{2\sqrt{x}}}_{u'(x)}$$

$$= \boxed{3x^2\cos x^6 - \frac{1}{2\sqrt{x}}\cos x}$$

May 16-9:47 AM

Find $\dfrac{d}{dx}\displaystyle\int_{2x}^{3x} \frac{t^2}{t^2+1}\,dt$

$v(x)$ $g(t)$ $u(x)$

$$= \frac{(3x)^2}{(3x)^2+1}\cdot 3 - \frac{(2x)^2}{(2x)^2+1}\cdot 2$$

$$= \frac{27x^2}{9x^2+1} - \frac{8x^2}{4x^2+1} = x^2\left[\frac{27(4x^2+1)-8(9x^2+1)}{(9x^2+1)(4x^2+1)}\right]$$

$$= \boxed{\frac{x^2(36x^2+19)}{(9x^2+1)(4x^2+1)}}$$

May 16-9:52 AM